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HOLOMORPHIC SECTIONAL AND BISECTIONAL CURVATURES OF ALMOST HERMITIAN MANIFOLDS

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Abstract. Friedland and Hsiung [1] proved an analogue of F. Schur's theorem concerning the holomorphic sectional curvature of some almost Hermitian manifolds called almost Hermitian L-manifolds, of which Kählerian manifolds are special ones. Recently, Hsiung and Xiong [3] gave a classification of almost Hermitian manifolds and extended the above work of Friedland and Hsiung to a new class of almost Hermitian manifolds called the class of almost C Hermitian manifolds.

In this paper we shall further extend the above work of Hsiung and Xiong by studying the general sectional, the holomorphic sectional and the holomorphic bisectional curvatures of almost Hermitian manifolds of all classes, together with some relationship among the three types of sectional curvatures.

§1. Introduction

Let M be a Riemannian $2n$-manifold, and g_{ij}, J_i^j and R_{hijk} the components of a Riemannian metric tensor, and an almost complex structure J, and the curvature tensor, of M respectively. Throughout this paper, all Latin indices take the values $1, ..., 2n$ unless stated otherwise. By using the following identities. Hsiung and Xiong [3] have defined the following four classes of almost complex structures on the Riemannian manifold M:

$$R_{hijk} = J_h{}^r J_i{}^s R_{rsjk}, \tag{1.1}$$

$$R_{hijk} = J_h^r J_i^s R_{rsjk} + J_h^r J_j^s R_{risk} + J_h^r J_k^s R_{rijs}, \tag{1.2}$$

$$R_{hijk} = J_h^p J_i^q J_j^r J_k^s R_{pqrs}, \tag{1.3}$$

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$$J_{i_1}{}^r J_{i_2}{}^s R_{rsi_3k} + J_{i_2}{}^r J_{i_3}{}^s R_{rsi_1k} + J_{i_3}{}^r J_{i_1}{}^s R_{rsi_2k} = 0, \tag{1.4}$$

where the repeated indices imply summation.

Let $\mathcal{L}$ and $\mathcal{K}$ denote the classes of almost complex structures (or manifolds) and the Kählerican structures (or manifolds), respectively. Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{C}$ denote the classes of almost complex structures (or manifolds) satisfying (1.1),...,(1.4) respectively. Hsiung and Xiong [3] have showed the following inclusion relation:

$$\mathcal{K} \subset \mathcal{L}_1 \subset \begin{matrix} \mathcal{L}_2 \\ \mathcal{C} \end{matrix} \subset \mathcal{L}_3 \subset \mathcal{L}. \tag{1.5}$$

Thus for $1 = 1, 2, 3$ as i decreases, the structures (or manifolds) in $\mathcal{L}_i$ resemble Kählerian structures (or manifolds) more closely.

If J_i^j and g_{ij} satisfy

$$g_{ij} J_h{}^i J_k{}^j = g_{hk}, \tag{1.6}$$

then the almost complex structure J and the manifold M are called an almost Hermitian structure and an almost Hermitian manifold, respectively, and g_{ij} is called an almost Hermitian metric. For simplicity, throughout this paper, unless stated otherwise, by an almost Hermitian manifold M we shall always mean a manifold with an almost Hermitian structure J and an almost Hermitian metric g_{ij}. Friedland and Hsiung [1] called an almost Hermitian structure J (or manifold M) an almost L structure (or manifold) if it satisfies

$$[\nabla_j, \nabla_k] J_i{}^h \equiv (\nabla_j \nabla_k - \nabla_k \nabla_j) J_i{}^h = 0, \tag{1.7}$$

where ∇ denotes the Levi-Civita connection of g_{ij}. Obviously, Kählerian manifolds are almost L manifolds since M is Kählerian if and only if

$$\nabla_i J_j{}^k = 0 \quad \text{for all} \quad i, j, k. \tag{1.8}$$

Friedland and Hsiung [1] have obtained a necessary and sufficient condition for an almost L manifold to have constant holomorphic sectional curvature H at each point and showed that H is an absolute constant for such a manifold. Hsiung and Xiong [3] have proved that an almost L manifold is an almost Hermitian $\mathcal{L}_1$ manifold and extended the above result of Friedland and Hsiung to an almost Hermitian $\mathcal{C}$ manifold.

The purpose of this paper is to extend further the above results of Hsiung and Xiong to an almost Hermitian manifold of each class with respect to a general sectional curvature or holomorphic sectional curvature, or holomorphic bisectional curvature, and to discuss the relationship among the three types of sectional curvatures for each of these manifolds.

In §2 (resp. §3) we recall some fundamental notation, definitions and well-known results on Riemannian structures (resp. almost complex structures) which are needed for the later discussions.

In §§ 4,5 and 6, we give a necessary and sufficient condition for an almost Hermitian manifold of each class to be of constant general sectional curvature, or constant holomorphic sectional curvature or constant holomorphic bisectional curvature at each point of the Riemannian manifold, respectively.

Some relationship among the three types of sectional curvatures for an almost Hermitian manifold of each class are derived in §7.

For simplicity we shall denote an almost Hermitian $\mathcal{L}_i$ manifold by AH_i for $i = 1, 2, 3$, and a Kählerian manifold, an almost Hermitian $\mathcal{C}$ manifold and an almost Hermitian manifold respectively by K, AHC and AH. From (1.5) we thus obtain the following inclusion relations

$$K \subset AH_1 \subset \begin{matrix} AH_2 \\ AHC \end{matrix} \subset AH_3 \subset AH. \tag{1.9}$$

Now we introduce the new notion of AH_1' manifold which denotes an almost Hermitian manifold satisfying

$$R_{hijk} = -J_h{}^r J_i{}^s R_{rsjk}. \tag{1.10}$$

It should be noted that the difference between (1.1) and (1.10) is only a sign, and therefore that $AH_1' \subset AHC \subset AH_3$, and the intersection of the two classes AH_1 and AH_1' is the class of locally Euclidean spaces, that is, the class of spaces with $R_{hijk} = 0$

§2. Riemannian structures

Let M be a Riemannian manifold of dimension $m \geq 2$ with Riemannian metric tensor g_{ij}, and let (g^{ij}) be the inverse matrix of (g_{ij}). We shall follow the usual tensor convention that indices can be raised and lowered by using g^{ij} and g_{ij} respectively. Let R_{hijk}, R_{ij}, R denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature of M, respectively.

The following identities are known, the last two of which are called the Bianchi identity and the Ricci identity respectively:

$$R_{hijk} + R_{hjki} + R_{hkij} = 0, \tag{2.1}$$

$$\nabla_\ell R_{hijk} + \nabla_j R_{hik\ell} + \nabla_k R_{hi\ell j} = 0, \tag{2.2}$$

$$\nabla_i \nabla_j T_k{}^h - \nabla_j \nabla_i T_k{}^h = T_k{}^s R^h{}_{sji} - T_s{}^h R^s{}_{kji}, \tag{2.3}$$

where ∇ denotes the Levi-Civita connection of M, and $T_k{}^h$ is an arbitrary tensor of type $(1,1)$.

The sectional curvature with respect to the two-dimensional plane (u,v) determined by two linearly independent tangent vectors u and v of M at a point p is given by

$$\begin{aligned} K = K(u,v) &= \frac{R_{hijk}u^h v^i u^j v^k}{(g_{hk}g_{ij} - g_{hj}g_{ik})u^h v^i u^j v^k} \\ &= \frac{R(u,v,u,v)}{[g(u,v)]^2 - g(u,u)g(v,v)}, \end{aligned} \tag{2.4}$$

where

$$R(u,v,u,v) = R_{hijk}u^k v^i u^j v^k, \tag{2.5}$$

$$g(u,u) = g_{ij}u^i u^j, \quad g(u,v) = g_{ij}u^i v^j, \quad g(u,v) = g_{ij}u^i v^j \tag{2.6}$$

Note that $K(u,v)$ is the Gaussian curvature of the two-dimensional geodesic submanifold of M tangent to the plane (u,v) at P. If the sectional curvature at any point of the Riemannian manifold does not depend on the two-dimensional plane at the point, then

$$R_{hijk} = K(g_{hk}g_{ij} - g_{hj}g_{ik}). \tag{2.7}$$

The Riemannian manifold is said to be locally Euclidean or locally flat if $K = 0$, i.e., if $R_{hijk} = 0$. For nonzero function K on M, from (2.2) and (2.7) it is easy to show that

$$R_{ij} = (m-1)Kg_{ij}, \tag{2.8}$$

$$R = m(m-1)K, \tag{2.9}$$

and for $m \geq 3$, K and therefore R are absolute constants on the manifold M and M is said to be of constant curvature. Furthermore, M is a n Einstein manifold as a consequence of (2.8).

Now we want to define an angle between 2-planes through a point p in the tangent space $T_p(M)$ of the Riemannian m-manifold M at p. Let $\Pi = (a,b)$ and $\Pi' = (c,d)$ be two 2-planes determined respectively by orthonormal tangent vectors a, b and c, d at the point p. Then the determinant

$$(\Pi, \Pi') = \begin{vmatrix} g(a,c) & g(a,d) \\ g(b,c) & g(b,d) \end{vmatrix} \tag{2.10}$$

is called the inner product of Π and Π'. When Π and Π' coincide, since a and b are not parallel, we have

$$(\Pi, \Pi) = g(a,a)g(b,b) - [g(a,b)]^2 > 0. \tag{2.11}$$

So we can define the angle $\langle \Pi, \Pi' \rangle$ between Π and Π' such that

$$\cos\langle \Pi, \Pi' \rangle = \frac{(\Pi, \Pi')}{\sqrt{(\Pi,\Pi)(\Pi',\Pi')}}, \quad 0 \leq \langle \Pi, \Pi' \rangle \leq \pi/2. \tag{2.12}$$

§3. Almost complex structures

In this section M is a Riemannian manifold as in § 2 but with dimension $m = 2n$. If a tensor $J_i{}^j$ of type (1.1) on M satisfies

$$J_i{}^j J_j{}^k = -\delta_i^k, \tag{3.1}$$

where δ_i^k are the Kronecker deltas defined by

$$\delta_i^k = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases}$$

then $J^{\,j_i}$ is called an almost complex structure on M, and M is called an almost complex manifold.

If J is Hermitian, then as a consequence of (3.1) and (1.6) the tensor J_{ij} of type $(0, 2)$ defined by

$$J_{ij} = g_{jk} J_i{}^k \tag{3.2}$$

is skew-symmetric. If the differential form J_{ij} is closed, then $J_i{}^j$ is called an almost Kählerian structure, and M an almost Kählerian manifold. It is clear that an almost Kählerian structure satisfies

$$J_{hij} \approx \nabla_h J_{ij} + \nabla_i J_{jh} + \nabla_j J_{hi} = 0. \tag{3.3}$$

The tensor J_{hij} is skew-symmetric in all indices.

An almost Hermitian structure $J_i{}^j$ satisfying

$$J_i \approx -\nabla_j J_i{}^j = 0 \tag{3.4}$$

is called an almost semi-Kählerian structure. An almost Kählerian structure is almost semi-Kählerian.

An almost Hermitian structure $J_i{}^j$ satisfying

$$\nabla_i J_j{}^k + \nabla_j J_i{}^k = 0 \tag{3.5}$$

is called a nearly Kählerian structure. Since $J_i{}^i = g^{ij} J_{ij} = 0$, from (3.4) and (3.5) it follows that *a nearly Kählerian manifold is almost semi-Kählerian.*

Let M be an almost Hermitian manifold with an almost complex structure $J_i{}^j$. Then the two-dimensional plane (u, Ju) determined by an arbitrary tangent vector u of M and the tangent vector Ju at a point p is called a holomorphic plane, and the sectional curvature with respect to the holomorphic plane at p is called the holomorphic sectional curvature at p. If the holomorphic sectional curvature at p is independent of the holomorphic plane at p, then M is said to be of constant holomorphic sectional curvature at p. We easily obtain

$$H(u) = K(u, Ju) = -\frac{R_{hpjq} J_i{}^p J_k{}^q u^h u^i u^j u^k}{[g(u,u)]^2} = -\frac{R(u, Ju, u, Ju)}{[g(u,u)]2}. \tag{3.6}$$

Let v be another tangent vector of M at p. Then the holomorphic bisectional curvature $B(u, v)$ of M at p with respect to the vectors u and v is defined as follows (see Goldberg and Kobayashi [2]):

$$B = B(u, v) = -\frac{R(u, Ju, v, Jv)}{g(u,u)g(v,v)}. \tag{3.7}$$

It is clear that $B(u, u) = H(u)$. So the holomorphic bisectional curvature is a generalization of the holomorphic sectional curvature.

Now let u and v be two unit tangent vectors of M at p, and let ϕ, θ and θ' be the angles between u and v, Ju and v and u and Jv, respectively. Then we obtain

$$\begin{aligned} \cos\phi &= g(u, v) = g(Ju, Jv) = g_{ij} u^i v^j, \\ \cos\theta &= g(Ju, v) = J_{ij} u^i v^j, \\ \cos\theta' &= g(Jv, u) = -\cos\theta. \end{aligned} \tag{3.8}$$

Furthermore, for two holomorphic planes $\Pi = (u, Ju)$ and $\Pi' = (v, Jv)$, we have, in consequence of (2.10),

$$\begin{aligned} (\Pi, \Pi) &= (\Pi'\Pi') = 1, \\ (\Pi, \Pi') &= \cos^2\phi + \cos^2\theta, \end{aligned} \tag{3.9}$$

which together with (2.12) imply

$$\cos\langle \Pi, \Pi' \rangle = \cos^2\phi + \cos^2\theta. \tag{3.10}$$

Thus

$$0 \le \cos^2\phi + \cos^2\theta \le 1. \tag{3.11}$$

In particular, when Π and Π' are orthogonal, $\phi = \theta = \pi/2$; when Π and Π' coincide, $\theta = \pi/2$ and $\phi \neq \theta$.

For the later developments, using (3.1), we can easily show that the following identities (3,12),(3,13),...,(3,16) are equivalent respectively to identities (2.1), (1.1),...,(1.4):

$$J_j{}^p J_k{}^q (R_{hipq} + R_{hpqi} + R_{hqip}) = 0, \tag{3.12}$$

$$J_j{}^p R_{hipk} + J_k{}^p R_{hijp} = 0, \tag{3.13}$$

$$J_h{}^p R_{pijk} + J_i{}^p R_{hpjk} + J_j{}^p R_{hipk} + J_k{}^p R_{hijp} = 0, \tag{3.14}$$

$$J_h{}^p J_i{}^q R_{pqjk} - J_j{}^p J_k{}^q R_{hipq} = 0, \tag{3.15}$$

$$J_h{}^p R_{kpij} + J_i{}^p R_{kpjh} + J_j{}^p R_{kphi} = 0. \tag{3.16}$$

§4. General sectional curvatures

In this section we shall discuss general sectional curvatures of almost Hermitian manifolds. At first we have

Theorem 4.1. *If an almost Hermitian $2n$-manifold M^{2n} is of constant general sectional curvature K at each point, then M^{2n} is an AH_3 manifold.*

Proof. The identity (2.7) yields

$$\begin{aligned} J_h{}^p J_i{}^q J_j{}^r J_k{}^s R_{pqrs} &= J_h{}^p J_i{}^q J_j{}^r J_k{}^s K(g_{ps}g_{qr} - g_{pr}g_{qs}) \\ &= K(J_{hs}J_{ir}J_j{}^r J_k{}^s - J_{hr}J_{is}J_j{}^r J_k{}^s) \\ &= K(g_{hk}g_{ij} - g_{hj}g_{ik}) = R_{hijk}, \end{aligned}$$

which is the defining equation (1.3) of an AH_3 manifold. □

Now we want to prove the following theorem for some smaller classes of almost Hermitian manifolds.

Theorem 4.2. *If an AH_1 or AH_1' $2n$-manifold M^{2n} for $n > 1$ has constant general sectional curvature K, then M^{2n} is locally flat.*

Proof. Suppose that M^{2n} is an AH_1 manifold of constant general sectional curvature K in (1.1), we can easily obtain

$$K(g_{hk}g_{ij} - g_{hj}g_{ik} + J_{hj}J_{ik} - J_{hk}J_{ij}) = 0. \tag{4.1}$$

Multiplying (4.1) by g_{ij} we have $(n-1)g_{hk}\ K = 0$, which implies $K = 0$.

Hence, by (2.7), $R_{hijk} = 0$ is deduced. Thus M^{2n} is locally flat.

If M^{2n} is an AH_1', manifold of constant general sectional curvature K, from (1.9), we have

$$K(g_{hk}g_{ij} - g_{hj}g_{ik} - J_{hj}J_{ik} + J_{hk}J_{ij}) = 0, \tag{4.2}$$

which yields $ng_{hk}K = 0$. Hence M^{2n} is also locally flat. □

§5. Holomorphic sectional curvatures

In this section we discuss holomorphic sectional curvatures of AH manifolds. For an AH manifold, Friedland & Hsiung [1] have established

Theorem 5.1. *A necessary and sufficient condition for an almost complex $2n$-manifold M^{2n} with an almost complex structure $J_i{}^j$ and a Riemannian metric g_{ij} to be of constant holomorphic sectional curvature H at each point is that the Riemann curvature tensor R_{hijk} of M^{2n} with respect to g_{ij} satisfies:*

$$\begin{aligned}(R_{jrsk}+R_{krsj})J_h{}^rJ_i{}^s &+ (R_{irsk}+R_{krsi})J_h{}^rJ_j{}^s \\ &+ (R_{irsj}+R_{jrsi})J_h{}^rJ_k{}^s + (R_{hrsk}+R_{krsh})J_i{}^rJ_j{}^s \\ &+ (R_{hrsj}+R_{jrsh})J_i{}^rJ_k{}^s + (R_{hrsi}+R_{irsh})J_k{}^rJ_j{}^s \\ &= 4H(g_{hi}g_{jk}+g_{ij}g_{hk}+g_{hj}g_{ik}).\end{aligned} \tag{5.1}$$

In this section, we use Theorem 5.1 to deduce a necessary and sufficient condition for an AH $2n$-manifold M^{2n} of each special class to have constant holomorphic sectional curvature H at each point. At first we have

Theorem 5.2. *A necessary condition for an AH $2n$-manifold M^{2n} to be of constant holomorphic sectional curvature H at each point is that the Riemann curvature tensor R_{hijk} of M^{2n} satisfies*

$$\begin{aligned}R_{hijk} = &-R_{hipq}J_j{}^pJ_k{}^q - R_{pqrs}J_h{}^pJ_i{}^qJ_j{}^rJ_k{}^s \\ &+ \frac{1}{3}P_{hijk} + \frac{1}{3}Q_{hijk} + \frac{4}{3}HG_{hijk},\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}P_{hijk} = &(R_{hpqk}J_i{}^q - R_{ipqk}J_h{}^q)J_j{}^p \\ &+ (R_{jpqk}-R_{jqpk})J_h{}^pJ_i{}^q \\ &+ (R_{jqpi}J_h{}^q - R_{jqph}J_i{}^q)J_k{}^p,\end{aligned} \tag{5.3}$$

$$\begin{aligned}Q_{hijk} = &-2R_{pqjh}J_h{}^pJ_i{}^q + (R_{hkpq}J_i{}^p - R_{ikpq}J_h{}^p)J_j{}^q \\ &+ (R_{jipq}J_h{}^q - R_{jhpq}J_i{}^q)J_k{}^p,\end{aligned} \tag{5.4}$$

$$G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik} + J_{hk}J_{ij} - J_{hj}J_{ik} - 2J_{hi}J_{jk}. \tag{5.5}$$

Furthermore, the Ricci tensor and scalar curvature of such a manifold are given respectively by

$$R_{ij} = -R_{pq}J_i{}^pJ_j{}^q + 3R_{piqr}J^{pq}J_j{}^r + 3R_{pjqr}J^{pq}J_i{}^r + 4(n+1)Hg_{ij}, \tag{5.6}$$

$$R = 3R_{psrq}J^{pq}J^{rs} + 4n(n+1)H. \tag{5.7}$$

Proof. Multiplying (5.1) by $J_{h_1}{}^{h}J_{i_1}^{i}$ and changing h_1, i_1 back to h, i respectively, we obtain

$$R_{hijk} + R_{hkji} = T_{hijk} - 4H(J_{ji}J_{jk} + g_{hj}g_{ik} + J_{hk}J_{ji}, \tag{5.8}$$

where

$$\begin{aligned} T_{hijk} = {} & (R_{hqpk} + R_{hkpq})J_i{}^pJ_j{}^q + (R_{hqpi} + R_{hipq})J_k{}^pJ_j{}^q \\ & + (R_{jqpk} + R_{jkpq})J_k{}^pJ_h{}^q + (R_{jqpi} + R_{jipq}J_k{}^pJ_i{}^h \\ & - (R_{spqr} + R_{srqp})J_h{}^pJ_i{}^qJ_j{}^rJ_k{}^s. \end{aligned} \tag{5.9}$$

Interchange of h and i in (5.8) yields

$$R_{ihjk} + R_{ikjh} = T_{ihjk} - 4H(J_{ih}J_{jk} + g_{ij}g_{hk} + J_{ik}J_{jh}). \tag{5.10}$$

Subtracting (5.10) from (5.8) and using (2.1) and (5.5) we easily have

$$3R_{hijk} = T_{hijk} - T_{ihjk} + 4H\ G_{hijk}. \tag{5.11}$$

On the other hand, from (5.9), (2.1), (5.3), (5.4) it follows that

$$\begin{aligned} T_{hijk} - T_{ihjk} = {} & -3R_{hipq}J_j{}^pJ_k{}^q - 3R_{pqrs}J_h{}^pJ_i{}^qJ_j{}^rJ_k{}^s \\ & + P_{hijk} + Q_{hijk}. \end{aligned} \tag{5.12}$$

Substitution of (5.12) in (5.11) gives immediately (5.2).

Multiplying (5.2) by g^{hk} and using (2.1), (3.4) we can obtain (5.6). Moreover, (5.7) follows similarly by multiplying (5.6) by g^{ij}.

The following theorem is a consequence of Theorem 5.2.

Theorem 5.3. *A necessary condition for an AH_3 $2n$-manifold M^{2n} to be of constant holomorphic sectional curvature H at each point is that the Riemann curvature tensor R_{hijk} of M^{2n} satisfies*

$$\begin{aligned} R_{hijk} = {} & \frac{2}{3}(R_{pqkj}J_h{}^pJ_i{}^q + R_{pkqi}J_h{}^pJ_j{}^q + R_{pjiq}J_h{}^pJ_k{}^q) \\ & - \frac{1}{3}(R_{pqjk}J_h{}^pJ_i{}^q + R_{piqk}J_h{}^pJ_j{}^q + R_{pijq}J_h{}^pJ_k{}^q) \\ & + \frac{2}{3}HG_{hijk}. \end{aligned} \tag{5.13}$$

Furthermore, the Ricci tensor and scalar curvature of such a manifold are given respectively by

$$R_{ij} = 3R_{piqr}J^{pq}J_j{}^r + 2(n+1)Hg_{ij}, \tag{5.14}$$

$$R = 3R_{psrq}J^{pq}J^{rs} + 4n(n+1)H. \tag{5.15}$$

Proof. Multiplying the defining equation (1.3) of an AH_3-manifold by $J_a{}^h J_b{}^i$ and $J_a{}^i J_b{}^k$, we obtain respectively

$$R_{rsjk}J_h{}^r J_i{}^s = R_{hirs}J_j{}^r J_k{}^s, \tag{5.16}$$

$$R_{hrsk}J_i{}^r J_j{}^s = R_{irsj}J_h{}^r J_k{}^s \tag{5.17}$$

Moreover, multiplying (5.17) by g^{hk} and g^{ik} gives respectively

$$R_{rs}J_i{}^r J_j{}^s = R_{ij}, \tag{5.18}$$

$$R_{piqr}J^{pq}J_j{}^r = R_{pjqr}J^{pq}J_i{}^r. \tag{5.19}$$

Using (5.17) and (2.1), we can reduce (5.3) and (5.4) respectively to

$$P_{hijk} = 2(R_{hpqk}J_i{}^q - R_{ipqk}J_h{}^q)J_j{}^p - R_{hipq}J_j{}^p J_k{}^q, \tag{5.20}$$

$$Q_{hijk} = -2R_{hipq}J_j{}^p J_k{}^q + 2(R_{hkpq}J_i{}^p - R_{ikpq}J_h{}^p)J_j{}^q. \tag{5.21}$$

Substitution of (1.3), (5.20), (5.21), (5.16) in (5.2) yields

$$\begin{aligned} 3R_{hijk} = &- 3R_{pqjk}J_h{}^p J_i{}^q + R_{hpqk}J_j{}^p J_i{}^q - R_{ipqk}J_j{}^p J_h{}^q \\ &+ R_{hkpq}J_i{}^p J_j{}^q - R_{ikpq}J_h{}^p J_j{}^q + 2HG_{hijk}. \end{aligned} \tag{5.22}$$

By means of (5.16), (5.17), (2.1) the first five terms on the right-hand side of (5.22) can be reduced to

$$\begin{aligned} &- 3R_{pqjk}J_h{}^p J_i{}^q + R_{pjiq}J_h{}^p J_k{}^q - R_{ipqk}J_j{}^p J_h{}^q \\ &+ (-R_{ipqj}J_h{}^p J_k{}^q + R_{ipqj}J_k{}^p J_h{}^q) \\ &- (-R_{ipqk}J_h{}^p J_j{}^q + R_{ipqk}J_j{}^p J_h{}^q). \end{aligned} \tag{5.23}$$

Substituting (5.23) in (5.22), we readily arrive at (5.13).

(5.14) can be obtained by multiplying (5.13) by g^{hk} and using (2.1), (5.19) or by applying (5.18), (5.19) to (5.6). Equation (5.15) follows immediately by multiplying (5.14) by g^{ij} $\square$

Now suppose that M^{2n} is an AH_3 Einstein manifold of constant holomorphic sectional curvature H. Since M^{2n} is an Einstein manifold, we have

$$R_{ij} = \frac{R}{2n} G_{ij}, \tag{5.24}$$

which together with (5.14) implies

$$R_{ij} = 3R_{piqr}J^{pq}J_j{}^r + 2(n+1)Hg_{ij} = \frac{R}{2n} g_{ij}, \tag{5.25}$$

so that $R_{ij} = \alpha R_{piqr}J^{pq}J_j{}^r = \beta H g_{ij}$, and therefore $R = \alpha R_{piqr}J^{pq}J^{ir} = 2n\beta H$. On an AH_1 manifold of constant holomorphic sectional curvature H, $\beta = \frac{n+1}{2}$[1], that is

$$H = \frac{R}{n(n+1)} \tag{5.26}$$

holds. We investigate an AH_3 manifold satisfying (5.26)

Lemma 5.4. *If M^{2n} is an AH_3 Einstein $2n$-manifold of constant nonzero holomorphic sectional curvature H with $H = \frac{R}{n(n+1)}$, then*

$$R_{ij} = \frac{n+1}{2} H g_{ij}, \tag{5.27}$$

$$R_{ij} = -R_{piqr}J^{pq}J_j{}^r, \tag{5.28}$$

$$Q := R + R_{piqr}J^{pq}J^{ir} = 0. \tag{5.29}$$

Proof. Equation (5.27) follows from (5.24) and (5.26). Substituting (5.26) in the right-hand side of (5.25), we obtain

$$(n+1)Hg_{ij} = -2R_{piqr}J^{pq}J_j{}^r. \tag{5.30}$$

Substitution of (5.30) in the left part of (5.25) gives (5.28) immediately. (5.29) is obtained by multiplying (5.28) by g^{ij}. $\square$

Corollary 5.5. *On a compact AH_3 Einstein $2n$-manifold of constant positive holomorphic sectional curvature H satisfying (5.26), the first Betti number of M^{2n} is zero.*

Proof. The result follows from (5.27) because the Ricci curvature tensor is positive definite [4].

It is known (see, for instance, [1]) that $Q \leq 0$ on an almost Kählerian manifold, and $Q \geq 0$ on a nearly Kählerian manifold. Furthermore, $Q = 0$ if and only if the manifold is Kählerian. Thus the following corollary is obvious due to (5.29).

Corollary 5.6. *An almost Kählerian or a nearly Kählelrian Einstein $2n$-manifold of constant nonzero holomorphic sectional curvature H satisfying (5.26) is Kählerian.*

In the following, we shall discuss Theorem 5.3 for three special AH_3 manifolds.

Theorem 5.7. *A necessary condition for an AH_2 $2n$-manifold M^{2n} to be of constant holomorphic sectional curvature H at each point is that the Riemann curvature tensor R_{hijk} satisfies*

$$\begin{aligned} R_{hijk} = &\frac{1}{2}(R_{pqkj}J_h{}^pJ_i{}^q + R_{pkqi}J_h{}^pJ_j{}^q + R_{pjiq}J_h{}^pJ_k{}^q) \\ &+ \frac{1}{2}HG_{hijk}. \end{aligned} \tag{5.31}$$

Furthermore, The Ricci tensor and scalar curvature of such a manifold are given respectively by

$$R_{ij} = 3R_{piqr}J^{pq}J_j{}^r + 2(n+1)Hg_{ij}, \tag{5.32}$$

$$R = 3R_{psrq}J^{pq}J^{rs} + 4n(n+1)H. \tag{5.33}$$

Proof. Substituting the defining equation (1.2) of an AH_2 manifold in the second term on the right-hand side of (5.13) we obtain (5.31) immediately. Multiplying (5.31) by g^{hk} and using (5.19), (5.23) is deduced. Furthermore, (5.33) follows by multiplying (5.32) by g^{ij}. □

Lemma 5.8. *An almost Kählerian or a nearly Kählerian AHC manifold M^{2n} is Kählerian.*

Proof. The defining equation (1.4) of an AHC manifold is equivalent to

$$R_{hijk} = R_{qhjp}J_i{}^pJ_k{}^q + R_{hqkp}J_i{}^pJ_j{}^q. \tag{5.34}$$

Multiplying (5.34) by $g^{hk}g^{ij}$ and using (2.1), we can easily obtain (5.29). Thus the lemma is proved in the same way as Corollary 5.6 was proved. □

Theorem 5.9. [3]. *A necessary condition for an AHC 2n-manifo ld M^{2n} to be of constant holomorphic sectional curvature H at each point is that the curvature tensor R_{hijk} satisfies*

$$R_{hijk} = R_{pqkj} J_h{}^p J_i{}^q + \frac{1}{2} H G_{hijk}. \tag{5.35}$$

Furthermore, the Ricci tensor and scalar curvature of such a manifold are given respectively by

$$R_{ij} = \frac{n+1}{2} H g_{ij}, \tag{5.36}$$

$$R = n(n+1)H. \tag{5.37}$$

As a consequence of (5.36), M^{2n} is an Einstein manifold. Furthermore, if M^{2n} is compact and H is positive, then the first Betti number of M^{2n} is zero.

Proof. Since an AHC manifold is AH_3, the curvature tensor R_{hijk} satisfies (5.13), which can be rewritten as

$$\begin{aligned} R_{hijk} = \frac{1}{3}[&4R_{pqkj} J_h{}^p J_i{}^q + R_{pqjk} J_j{}^p J_i{}^q + (R_{pikq} J_h{}^p J_i{}^q + R_{pkqi} J_h{}^p J_j{}^q) \\ &+ (R_{piqj} J_h{}^p J_k{}^q + R_{pjiq} J_h{}^p J_j{}^q)(R_{pkqi} J_h{}^p J_j{}^q + R_{pjiq} J_h{}^p J_k{}^q) \\ &+ H G_{hijk}]. \end{aligned} \tag{5.38}$$

By (3.17), (5.38) is reduced to

$$\begin{aligned} R_{hijk} = \frac{1}{3}[&4R_{pqkj} J_h{}^p J_i{}^q + (R_{pqjk} J_h{}^p J_i{}^q + R_{pqki} J_h{}^p J_j{}^q + R_{pqij} J_h{}^p J_k{}^q) \\ &+ (R_{pkqi} J_h{}^p J_j{}^q + R_{pjiq} J_h{}^p J_k{}^q) + 2H G_{hijk}], \end{aligned} \tag{5.39}$$

which together with (3.21) and (5.34) gives (5.35).

By multiplying (5.34) by G^{hk} and using (5.18) we can easily obtain

$$R_{ij} = \frac{1}{2} R_{qhjp} J^{hq} J_i{}^p. \tag{5.40}$$

On the other hand, from (3.8) it follows that

$$R_{pqkj} J_h{}^p J_i{}^q = -R_{pkjq} J_i{}^q J_h{}^p + R_{pjkq} J_i{}^q J_h{}^p. \tag{5.41}$$

Multiplying (5.41) by g^{hk}, we have

$$R_{pqkj} J^{kp} J_i{}^q = -\frac{1}{2} R_{pkjq} J^{kp} J_i{}^q. \tag{5.42}$$

Multiplying (5.35) by g^{hk} using (5.42) and (5.40), we thus arrive at (5.36).

Equation (5.37) is obvious, and the last part of this theorem follows from the same argument as in the proof of Corollary 5.5. □

Theorem 5.10 [1]. *A necessary and sufficient condition for an AH_1 2n-manifold M^{2n} to be of constant holomorphic sectional curvature H at each points is that the curvature tensor R_{hijk} satisfies*

$$R_{hijk} = \frac{1}{4} H G_{hijk}. \tag{5.43}$$

Furthermore, the Ricci tensor and scalar curvature of such a manifold are respectively given by (5.36), (5.37). As a consequence of (5.36), M^{2n} is an Einstein manifold. Furthermore, if M^{2n} is compact and H is positive, then the first Betti number of M^{2n} is zero.

Proof. (5.43) is a consequence of (5.35), (1.1), (1.5).

For the sufficiency of the theorem, we notice that from (3.11) it follows that M^{2n} has constant holomorphic sectional curvature H is and only if

$$R_{risk} J_h{}^r u^h J_j{}^s u^j u^k = -H g_{hi} u^h u^i g_{jk} u^j u^k \tag{5.44}$$

holds for any tangent vector u^i of M^{2n}. If (5.43) holds, then by substituting (5.43) in the left-hand side of (5.44), we can easily show that the left-hand side of (5.44) becomes automatically the right-hand side of (5.44).

The other part of the theorem follows from the same argument as in the proof of Theorem 5.5. □

§6. Holomorphic bisectional curvatures

This section is devoted to a study of the holomorphic bisectional curvatures of AH manifolds. At first we have

Theorem 6.1. *A necessary and sufficient condition for an AH 2n-manifold M^{2n} to be of constant holomorphic bisectional curvature B at each point is that the Riemann curvature tensor R_{hijk} satisfies*

$$\begin{aligned} R_{hpjq} J_i{}^p J_k{}^q + R_{ipjq} J_h{}^p J_k{}^q + R_{hpkq} J_i{}^p J_j{}^q \\ + R_{ipkq} J_h{}^p J_j{}^q = -4B g_{hi} g_{jk}. \end{aligned} \tag{6.1}$$

Proof. To prove the necessity of condition (6.1) we assume that M^{2n} is of constant holomorphic bisectional curvature B. Then, from (3.7) it follows that

$$R_{hpiq} J_i{}^p J_k{}^q u^h u^i v^j v^k = -B g_{hi} g_{jk} u^h u^i v^j v^k \tag{6.2}$$

for any tangent vectors u and v of M^{2n}. By collecting all the coefficients of a general term $u^h u^i v^j v^k$ on the left-hand side of (6.2) by interchanging the indices h, i, j, k in all possible cases, i.e., interchanging h, i and keeping j, k, interchanging j, k and keeping h, i and interchanging h, i and interchanging j, k at the same time, we can easily obtain the left-hand side of (6.1). In the same way we can show that all the coefficients of the general term $u^h u^i v^j v^k$ on the right-hand side of (6.2) is the right-hand side of (6.1).

To prove the sufficiency of condition (6.1) we suppose that (6.1) holds. Multiplying both sides of (6.1) by $u^h u^i v^j v^k$ for any tangent vectors u and v of M^{2n} and summing for h, i, j, k we can see that all the terms on the left-hand side of the resulting equation are equal to each other. Thus (6.2) holds for any tangent vectors u and v of M^{2n}, that is, M^{2n} is of constant holomorphic bisectional curvature at each point. Hence the proof of this theorem is complete.

Theorem 6.2. *A necessary and sufficient condition for an AH 2n-manifold M^{2n} to be of constant holomorphic bisectional curvature B with respect to g_{ij} at each point is that the Riemann curvature tensor R_{hijk} satisfies*

$$\begin{aligned} -R_{hijk} = {} & R_{pqrs} J_h{}^p J_i{}^q J_j{}^r J_k{}^s + R_{pqjk} J_h{}^p J_i{}^q \\ & + R_{hipq} J_j{}^p J_k{}^q + 4B J_{hi} J_{jk}. \end{aligned} \tag{6.3}$$

Furthermore the Ricci tensor and the scalar curvature of such a manifold are given respectively by

$$R_{ij} = R_{pjqr} J_i{}^r J^{pq} + R_{piqr} J_j{}^r J^{pq} - R_{pq} J_i{}^p J_j{}^q + 4B g_{ij}, \tag{6.4}$$

$$R = R_{prqs} J^{pq} J^{rs} + 4nB. \tag{6.5}$$

Proof. To prove the necessity of condition (6.3), we suppose that M^{2n} is of constant holomorphic bisectional curvature B. Then, by Theorem 6.1, we have (6.1). Multiplying (6.1) by $J_r{}^i J_s{}^k$, (6.3) is immediately deduced. Also, we obtain (6.4) by multiplying (6.3) by g^{hk}, and obtain (6.5) by multiplying (6.4) by g^{ij}.

To prove the sufficiency of condition (6.3) we suppose that (6.3) holds. Multiplying (6.3) by $J_a{}^i J_b{}^k$, (6.1) is readily obtained. Thus M^{2n} is of constant holomorphic bisectional curvature at each point by Theorem 6.1. □

The following corollary is an obvious consequence of (6.1) and (6.3).

Corollary 6.3. *A necessary and sufficient condition for an AH 2n-manifold M^{2n} to be of zero holomorphic bisectional curvature at each point is that the Riemann curvature tensor R_{hijk} satisfies*

$$\begin{aligned} & R_{hpjq} J_i{}^p J_k{}^q + R_{ipjq} J_h{}^p J_k{}^q + R_{hpkq} J_i{}^p J_j{}^q \\ & + R_{ipkq} J_h{}^p J_j{}^q = 0, \end{aligned} \tag{6.6}$$

or

$$\begin{aligned} -R_{hijk} = {} & R_{pqrs}J_h{}^pJ_i{}^qJ_j{}^rJ_k{}^s + R_{pqjk}J_h{}^pJ_i{}^q \\ & + R_{hipq}J_j{}^pJ_k{}^q. \end{aligned} \tag{6.7}$$

Remark. From (1.9) and (6.7) it follows immediately that an AH_1' manifold has zero holomorphic bisectional curvature at each point.

Theorem 6.4. *A necessary and sufficient condition for an AH_3 2n-manifold M^{2n} to be of constant holomorphic bisectional curvature B at each point is that the Riemann curvature tensor R_{hijk} satisfies*

$$-R_{hijk} = R_{hipq}J_j{}^pJ_k{}^q + 2BJ_{hi}J_{jk}. \tag{6.8}$$

Furthermore, the Ricci curvature and the scalar curvature of such a manifold are given, respectively, by

$$R_{ij} = R_{hiqp}J_j{}^pJ^{hq} + 2Bg_{ij}, \tag{6.9}$$

$$R = R_{hiqp}J^{ip}J^{hq} + 4nB. \tag{6.10}$$

Proof. By means of (1.3), we can see that on the right-hand side of (6.3) the first term is R_{hijk}, and the second and third terms are the same, so that (6.3) becomes (6.8). Multiplying (6.8) by g^{hk}, (6.9) is deduced and, furthermore, (6.10) is obtained by multiplying (6.9) by g^{ij}. □

Corollary 6.5. *A necessary and sufficient condition for an AH_3 manifold M^{2n} to be of zero holomorphic bisectional curvature is that M^{2n} is an AH_1' manifold.*

Proof. This follows immediately from (6.8) and (1.9) □

Theorem 6.6. *A necessary and sufficient condition for an AH_2 2n-manifold M^{2n} to be of constant holomorphic bisectional curvature B at each point is that the Riemann curvature tensor R_{hijk} satisfies*

$$R_{hijk} = R_{piqk}J_h{}^pJ_j{}^q + B(g_{hi}g_{jk} - J_{hi}J_{jk}). \tag{6.11}$$

Furthermore, the Ricci curvature and the scalar curvature of such a manifold are given, respectively, by

$$R_{ij} = R_{piqk}J^{pq}J_j{}^k + 2Bg_{ij}, \tag{6.12}$$

$$R = R_{piqk}J^{pq}J^{ik} + 4nB. \tag{6.13}$$

Proof. Since M^{2n} is an AH_3 manifold by (1.9), (6.8) holds. By multiplying (6.8) by $J_\ell{}^k$, we obtain

$$R_{p\ell hi}J_j{}^p - R_{jphi}J_\ell{}^p = 2BJ_{hi}g_{j\ell},$$

which becomes, after some changes of indices,

$$R_{pijk}J_h{}^p - R_{hpjk}J_i{}^p = 2BJ_{jk}g_{hi}. \tag{6.14}$$

Similarly, we have

$$R_{pkhi}J_j{}^p - R_{jphi}J_k{}^p = 2BJ_{hi}g_{ik}. \tag{6.15}$$

Subtracting (6.14) and (6.15) from (3.10), which holds for an AH_2 manifold, we have

$$R_{hpjk}J_i{}^p + R_{jphi}J_k{}^p = -B(J_{ik}g_{hi} + J_{hi}g_{jk}). \tag{6.16}$$

By multiplying (6.16) by $J_q{}^i$, we arrive at (6.11).

Multiplying (6.11) by g^{hk}, (6.12) is deduced, and, furthermore, (6.13) is obtained by multiplying (6.12) by g^{ij}. □

Theorem 6.7. *If an AHC 2n-manifold M^{2n} is of constant holomorphic bisectional curvature B, then B must be zero and M^{2n} is an AH_1' manifold.*

Proof. Since M^{2n} is an AH_3 manifold by (1.9), (6.14) holds. By changing the subscripts i, j, k cyclically from (6.14), we obtain two more equations. On the left-hand side of these three equations, the sum of the three first terms is zero by (2.1), and the sum of the three second terms is zero by (3.12), so that we obtain

$$2B(J_{ik}g_{hi} + J_{ki}g_{hj} + J_{ij}g_{hk}) = 0,$$

which implies $B = 0$, and hence M^{2n} is an AH_1' by Corollary 6.5. □

§7. The relationship among the three types of sectional curvatures

In this section we shall assume, unless stated otherwise, that M is an almost Hermitian $2n$-manifold with an almost Hermitian structure J and an almost Hermitian metric g whose respective components are $J_i{}^j$ and g_{ij}. Moreover, let u and v be two unit tangent vectors of M at a point p, and let ϕ, θ, θ' be the angles between u and v, Ju and v, u and Jv, respectively. Then, from (2.4) and (3.13) it follows that the sectional curvature of M with respect to the two- dimensional plane (u, v) determined by two linearly independent unit tangent vectors u and v at p is given by

$$K(u, v) = -R(u, v, u, v)\sin^{-2}\phi. \tag{7.1}$$

Theorem 7.1. *If M is an AH_1 manifold, then the sectional curvature $K(u,v)$ and the holomorphic bisectional curvature $B(u,v)$ satisfy*

$$B(u,v) = K(u,v)\sin^2\phi + K(u,Jv)\sin^2\theta.$$

Proof. From (1.1), (2.1), (3.8) and (7.1) we obtain

$$\begin{aligned} B(u,v) &= -R_{hijk}u^h J_p{}^i u^p v^j J_q{}^k v^q \\ &= (R_{hjki} + R_{hkij})u^h J_p{}^i u^p v^j J_q{}^k v^q \\ &= -R_{hjik}u^h v^j J_p{}^i u^p J_q{}^k v^q \\ &\quad + R_{hkij}u^h J_q{}^k v^q J_p{}^i u^p v^j \\ &= -(R_{hjpq}J_i{}^p J_k{}^q)u^h v^j u^i v^k \\ &\quad - (R_{hkpq}J_i{}^p J_j{}^q)u^h J_r{}^k v^r u^i J_s{}^j v^s \\ &= -R_{hjik}u^h v^j u^i v^k - R_{hkij}u^h J_r{}^k v^r u^i J_s{}^j v^s \\ &= K(u,v)\sin^2\phi + K(u,Jv)\sin^2\theta. \end{aligned} \tag{7.3}$$

Corollary 7.2. *Assume that M is an AH_1-manifold. If the two holomorphic planes $\Pi = (u, Ju)$ and $\Pi' = (u, Jv)$ are orthogonal, then*

$$B(u,v) = K(u,v) + K(u,Jv). \tag{7.4}$$

Proof. This result is immediately deduced from the previous theorem and (3.10). $\square$

For a Kählerian manifold, Goldberg and Kobayashi obtained (7.4) in [2], but missed the orthogonality condition of the two holomorphic planes (u, Ju) and (v, Jv).

Corollary 7.3. *If M is an AH_1' manifold, then the sectional curvature $K(u,v)$ and the holomorphic bisectional curvature $B(u,v)$ satisfy*

$$-B(u,v) = K(u,v)\sin^2\phi + K(u,Jv)\sin^2\theta. \tag{7.5}$$

In particular, if (u, Ju) and (v, Jv) are orthogonal, then

$$-B(u,v) = K(u,v) + K(u,Jv). \tag{7.6}$$

Proof. This corollary is deduced by imitating the proof of Theorem 7.1. $\square$

If an almost Hermitian manifold M has constant general sectional curvature, then from the definition M must also have constant holomorphic sectional curvature, but the following theorem shows that M does not necessarily have constant holomorphic bisectional curvature.

Theorem 7.4. *If an almost Hermitian manifold M has nonzero constant general sectional curvature K, then the holomorphic bisectional curvature $B(u,v)$ of M satisfies*

$$B(u,v) = K(\cos^2\phi + \cos^2\theta) = K\cos\langle\Pi,\Pi'\rangle, \tag{7.7}$$

where $\langle\Pi,\Pi'\rangle$ is the angle between the two planes $\Pi := (u, Ju)$ and $\Pi' := (v, Jv)$ defined by (2.12).

Proof. By (3.7) and (2.7), we have

$$B(u,v) = -K(g_{hk}g_{ij} - g_{hj}g_{ik})u^h J_p{}^i u^p v^j J_q{}^k v^q.$$

Substitution of (3.8) and (3.10) in the above equation gives immediately (7.7). □

The following two corollaries are immediate consequences of Theorem 7.4 and (3.11).

Corollary 7.5. *For an almost Hermitian manifold M with nonzero constant general sectional curvature K, the holomorphic bisectional curvature $B(u,v)$ has the same sign as K and*

$$\begin{aligned} 0 \le B(u,v) \le K \qquad &\text{for} \quad K > 0,\\ K \le B(u,v) \le 0 \qquad &\text{for} \quad K < 0. \end{aligned} \tag{7.8}$$

Moreover, the absolute value $|B(u,v)|$ reaches zero, the minimum, when the two holomorphic planes $\Pi = (u, Ju)$ and $\Pi' = (v, Jv)$ are orthogonal, and reaches $|K|$ when Π and Π' coincide.

Corollary 7.5. *A necessary and sufficient condition for two holomorphic planes $\Pi = (u, Ju)$ and $\Pi' = (v, Jv)$ of an almost Hermitian manifold M with nonzero constant general sectional curvature to be orthogonal is that the holomorphic bisectional curvature of M determined by Π and Π' is zero.*

Theorem 7.7. *Let M be an AH_1 manifold with nonzero constant holomorphic sectional curvature H. Then the general sectional curvature $K(u,v)$ and the holomorphic bisectional curvature $B(u,v)$ are*

$$K(u,v) = \frac{H}{4}\left(1 + \frac{3\cos^2\theta}{\sin^2\phi}\right), \tag{7.9}$$

$$B(u,v) = H\cos^2\frac{\langle\Pi,\Pi'\rangle}{2}, \tag{7.10}$$

respectively.

Proof. By (7.1), (5.43), (5.5) and (3.8) we obtain

$$
\begin{aligned}
-K(u,v)\sin^2\phi &= R_{hijk}u^h v^i u^j v^k \\
&= \frac{1}{4}HG_{hijk}u^h v^i u^j v^k \\
&= -\frac{H}{4}(\sin^2\phi + 3\cos^2\theta), \qquad (7.11)
\end{aligned}
$$

which implies (7.9).

Similarly, we have, in consequence of (3.7) and (3.10),

$$
\begin{aligned}
-B(u,v) &= R_{hijk}u^h J_p{}^i u^p v^j J_q{}^k v^q \\
&= -\frac{1}{4}HG_{hijk}u^h J_p{}^i u^p v^j J_q{}^k v^q \\
&= H\cos^2\frac{\langle \Pi, \Pi' \rangle}{2}, \qquad (7.12)
\end{aligned}
$$

which implies (7.10). □

The following corollary is an immediate consequence of (7.9).

Corollary 7.8. *Using the same notation as in Theorem 7.7 for orthogonal vectors u and v we have*

$$
\begin{aligned}
&\frac{H}{4} \le K \le H \qquad \text{for} \quad H > 0, \\
&\frac{H}{4} \ge K \ge H \qquad \text{for} \quad H < 0. \qquad (7.13)
\end{aligned}
$$

In Corollary 7.8, $\frac{H}{4} = K$ occurs when Ju and v are orthogonal, and $H = K$ occurs when Ju and v coincide.

Since a Kählerian manifold is an AH_1 manifold, the following corollary is obvious.

Corollary 7.9. *Both Theorem 7.7 and Corollary 7.8 are true for a Kählerian manifold.*

For (7.9) and Corollary 7.8 for a Kählerian manifold see Kon and Yano [5, pp. 76–77].

Corollary 7.10. *Any AH_1 or Kählerian $2n$-manifold M for $n \ge 2$ of constant holomorphic bisectional curvature is locally flat.*

Proof. Since M is of constant holomorphic bisectional curvature B, every holomorphic sectional curvature $H(u,v)$ is constant. Suppose $H \ne 0$. Then from (7.12), $\cos^2(\langle \Pi, \Pi' \rangle/2)$ is constant, which is impossible for $n \ge 2$. Thus $H = 0$ which implies that $R_{hijk} = 0$ by (5.43). Hence M is locally flat. □

For AHC manifolds, which are more general than AH_1 manifolds, of nonzero constant holomorphic sectional curvature, we have the following theorem.

Theorem 7.11. *If M is an AHC manifold of nonzero constant holomorphic sectional curvature H, then the general sectional curvature K and the holomorphic bisectional curvature B of M satisfy*

$$K(u,v)\sin^2\phi + K(u,Jv)\sin^2\theta = H\cos^2\frac{\langle\Pi,\Pi'\rangle}{2}, \tag{7.14}$$

$$B(u,v) = H\cos^2\frac{\langle\Pi,\Pi'\rangle}{2}. \tag{7.15}$$

Proof. It is known [3] that the Riemann curvature tensor R_{hijk} of M satisfies

$$R_{hijk} = R_{hpqj}J_i{}^pJ_k{}^q - \frac{1}{2}H(g_{hj}g_{ik} + g_{hi}g_{jk} + J_{hj}J_{ik} + J_{hi}J_{jk}). \tag{7.16}$$

Substituting (7.14) in (7.11) and using (3.8), (3.10) and (7.11) for $K(u,Jv)$, we obtain

$$\begin{aligned} K(u,v)\sin^2\phi &= R_{hrjs}J_i{}^rJ_k{}^su^hv^iu^jv^k \\ &\quad + \frac{1}{2}H(g_{hj}g_{ik} + g_{hi}g_{jk} + J_{hj}J_{ik} + J_{hi}J_{jk})u^hv^iu^jv^k \\ &= R_{hijk}u^hJ_p{}^iv^pu^jJ_q{}^kv^q \\ &\quad + \frac{1}{2}H(1+\cos^2\phi + \cos^2\theta) \\ &= -K(u,Jv)\sin^2\theta + \frac{1}{2}H(1+\cos\langle\Pi,\Pi'\rangle), \end{aligned}$$

which implies (7.14). Similarly, from (7.3) it follows that

$$\begin{aligned} B(u,v) &= J_i{}^rJ_k{}^sR_{hrjs}u^hJ_p{}^iu^pv^jJ_q{}^kv^q \\ &\quad + \frac{1}{2}H(g_{hj}g_{ik} + g_{hi}g_{jk} + J_{hj}J_{ik} + J_{hi}J_{jk})u^hJ_p{}^iu^pv^jJ_q{}^kv^q \\ &= R_{hrjs}u^hu^rv^jv^s + \frac{1}{2}H(1+\cos^2\phi + \cos^2\theta) \\ &= \frac{1}{2}H(1+\cos\langle\Pi,\Pi'\rangle), \end{aligned}$$

which implies (7.15) □

Corollary 7.12. *If an AHC manifold M is of constant holomorphic sectional curvature H, then the sectional curvature K and the holomorphic bisectional curvature B satisfy*

$$B(u,v) = K(u,v)\sin^2\phi + K(u,Jv)\sin^2\theta. \tag{7.17}$$

Proof. The corollary follows immediately from (7.14) and (7.15). □

Remark. Theorem 7.1 shows that (7.17) also holds for an AH_1 manifold M, but there is no constant holomorphic sectional curvature condition on M.

Corollary 7.13. *If M is an AHC manifold of nonzero constant holomorphic sectional curvature H, then the holomorphic bisectional curvature B of M satisfies*

$$\frac{H}{2} \leq B(u,v) \leq H, \qquad \textit{for} \quad H > 0,$$
$$H \leq B(u,v) \leq \frac{H}{2}, \qquad \textit{for} \quad H < 0.$$

In Corollary 7.13, $\frac{H}{2} = B(u,v)$ occurs when $\Pi = (u, Ju)$ and $\Pi'(v, Jv)$ are orthogonal, and $H = B(u,v)$ occurs when Π and Π' coincide.

References

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